Simplified Calculation of Bhattacharyya Parameters in Polar Codes

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***Abstract*—The construction of polar code refers to selecting K "most reliable polarizing channels" in N polarizing channels $W_N^{(i)}$ to transmit information bits. For non-systematic polar code, Arikan proposed a method to measure the channel reliability for BEC channel, which is called Bhattacharyya Parameter method. The calculated complexity of this method is $O(N)$. In this paper, we find the complementarity of Bhattacharyya Parameter. According to the complementarity, the code construction under a certain channel condition can be quickly deduced from the complementary channel condition.**

***Keywords*—Reliable channel; Bhattacharyya parameter; binary erasure channel**

I. Introduction

Polar codes are the first proven to achieve channel capacity over various binary-input discrete memoryless channels (B-DMCs) [1]. Polar codes have regular, low complexity coding and decoding structure. Furthermore, when the code length is N, it was shown by Arikan and Telatar [2] that the block error probability of polar codes is $O(2^{-N^{\beta}})$ for any fixed $\beta < \frac{1}{2}$. It is significantly fast since the block error probability of low-density parity-check (LDPC) codes is polynomial in N [3].

An important issue in polar codes is how to efficiently identify the positions and rank these good and bad channels according to reliability, which is called polar code construction. Up to now, efficient construction is to calculate Bhattacharyya Parameter for binary erasure channel (BEC). It is shown by Mari[4] that complexity of code construction is $O(N)$ for arbitrary symmetric binary-input memoryless channel (B-MC).In the case of AWGN channel, several techniques are commonly used, such as density evolution (DE) and Gaussian approximation (GA) of density evolution [5]. In this paper, the inherent law of Bhattacharyya parameter is discovered, which can be used to simplify calculation. Furthermore, we can calculate Bhattacharyya parameter under complementary channel erasure probability easily.

For the AWGN channel, no low-complexity code construction algorithm has yet been found. However we have known that the reliability of the symmetric channel has a certain order relationship [6]-[8]. In addition, using this order relationship, we can more simply calculate the reliability of the symmetric channel [9].

The paper is organized in the following sections. In section II, we review some basic definitions. In section III, the complementarity of the Bhattacharyya parameter is described in detail, and the application of complementarity in calculating the Bhattacharyya Parameter is analyzed. Finally, this paper is concluded in Section IV.

II. Preliminaries

A. Channel Polarization

Channel polarization is the core theory in polar codes. It includes "channel combining" and "channel splitting". Channel combining means N copies of channel W are combined into W_N. In fact, channel combining reflects the coding process.

$$x_1^N = u_1^N * G_N \tag{1}$$

where x_i represents the code word after encoding, the u_i represents the input of channel and G_N represents generator matrix. The transition probability of the combining channel W_N is

$$W_N(y_1^N|u_1^N) = W^N(y_1^N|u_1^N * G_N) \tag{2}$$

Channel splitting means splitting the channel $\mathcal{W}: \mathcal{X} \to \mathcal{Y}$ into a pair of channels $\mathcal{W}^0: \mathcal{X} \to \mathcal{Y}^2$ and $\mathcal{W}^1: \mathcal{X} \to \mathcal{X} \times \mathcal{Y}$. We can obtain different channels$\{W_N^0, W_N^1, \dots W_N^{N-1}\}$ from the original channel W when the code length is N. The transition probability of channel splitting is defined as

$$W_N^{(i)}(y_1^N, u_1^{i-1}|u_i) = \sum_{u_{i+1}^N \in \mathcal{X}^{N-i}} \frac{1}{2^{N-1}} W_N(y_1^N|u_1^N) \tag{3}$$

where u_i denotes the input of $W_N^{(i)}$ and (y_1^N, u_1^{i-1}) denotes the output of $W_N^{(i)}$ .The notation $v_i^j (i \leq j)$ denotes the vector $(v_i, \dots v_j)$. Channel polarization process is shown in Fig. 1

B. Bhattacharyya Parameter

The Bhattacharyya parameter refers to the upper limit of the maximum likelihood decision error probability when the

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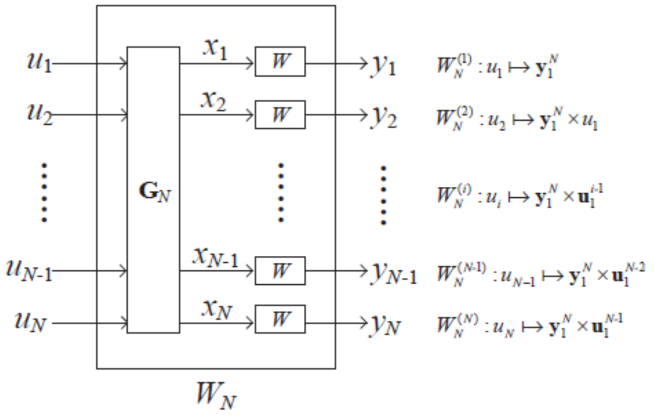

Figure 1. Channel combining and splitting

channel only transmits 0 or 1. It is a measure of channel reliability. For BEC,

$$\mathrm{Z(W)} = \sum_{y\in\mathcal{Y}} \sqrt{W(y|0)W(y|1)} \tag{4}$$

Smaller values mean more reliable the channel is. The Bhattacharyya parameter of the ith virtual channel $W_N^{(i)}$ is defined as

$$\mathrm{Z}\left(W_N^{(i)}\right) = \sum_{y_1^N\in\mathcal{Y}^N} \sum_{u_1^{i-1}\in\mathcal{X}^{i-1}} \sqrt{W_N^{(i)}(y_1^N, u_1^{i-1}|0) W_N^{(i)}(y_1^N, u_1^{i-1}|1)} \tag{5}$$

As $\mathrm{N}\to\infty$, $\mathrm{Z}(W_N^{(i)}) \to 0$ or $\mathrm{Z}(W_N^{(i)}) \to 1$. $\mathrm{Z}(W_N^{(i)}) \to 0$ means ith virtual channel is noiseless. On the contrary, $\mathrm{Z}(W_N^{(i)}) \to 1$ means ith virtual channel is fully noisy channel. This reflects the channel polarization.

C. *The Construction of Polar Codes*

The construction of polar codes refers to choosing K "most reliable polarizing channels" in N polarizing channels $W_N^{(i)}$ to transmit information bits. The method to measure channel reliability is to calculate Bhattacharyya parameter for BEC. So, the method of construction for BEC is selecting the K polarization channels with the smallest Bhattacharyya Parameters values to transmit information bits and selecting others to transmit frozen bits.

III. Complementarity of Bhattacharyya Parameters Based on Erasure Probability

A. *Variation of Bhattacharyya Parameters*

The calculation formula of Bhattacharyya parameter for BEC is defined as

$$\begin{cases} Z_w^1 = 2\times Z_w - (Z_w)^2 \\ Z_w^2 = (Z_w)^2 \end{cases} \tag{6}$$

Explanation of formula (6): As mentioned above, the Bhattacharyya parameter refers to the upper limit of the maximum likelihood decision error probability. So, we can obtain that the Bhattacharyya parameter is equal to the channel erasure probability for BEC. Therefore, we can get the relationship between channel capacity and Bhattacharyya parameter as follows:

$$I(W_N^i) = 1 - Z(W_N^i) \tag{7}$$

where $I(W_N)$ represents channel capacity. We can obtain formula (6) according to channel capacity calculation rules for BEC [1].

We can draw conclusions from formula (6) that Bhattacharyya parameters split into two values after a polarization process, one is bigger, the other is smaller. But the variation is the same, it is $Z_W - (Z_W)^2$ and the variation corresponds to the value of Bhattacharyya parameter.

Theorem 1: y represents the variation after one polarization process; x represents the current Bhattacharyya parameter value, then

$$\mathrm{y} = \mathrm{x} - x^2 \tag{8}$$

It is simple to be proved according to (6). In addition,

$$\mathrm{y(0.5+x)} = \mathrm{y(0.5-x)} \tag{9}$$

it means the symmetry axis of the function is $x = 0.5$.The function diagram is shown in Fig.2; Variation reaches the maximum 0.25 when Bhattacharyya parameters is 0.5

B. *Binary Representation of Input Bits*

Polar code is multi-stage coding. There are total of $\log_2 N$ coding stages, and each stage has $\mathrm{N}/2$ polarization units. For example, the coding process with $N = 8$ is shown in Fig.3.

We give the following definition:

process 0: the polarization process through upper branch of the polarization unit and the value of the Bhattacharyya parameter increased;

process 1: the polarization process through lower branch of the polarizing unit and the value of the Bhattacharyya parameter reduced.

Then the input bits $u_i(i_1 \dots i_{n-1} i_n)$ can be uniquely represented by “process 0” or “process 1”. And i_n represents the process of nth stage. For example, from Fig.3, we can get that,

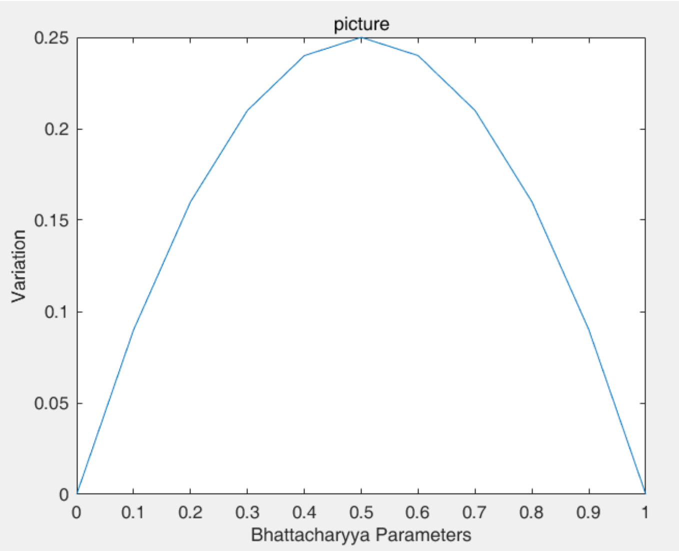

Figure 2. $y = x - x^2$

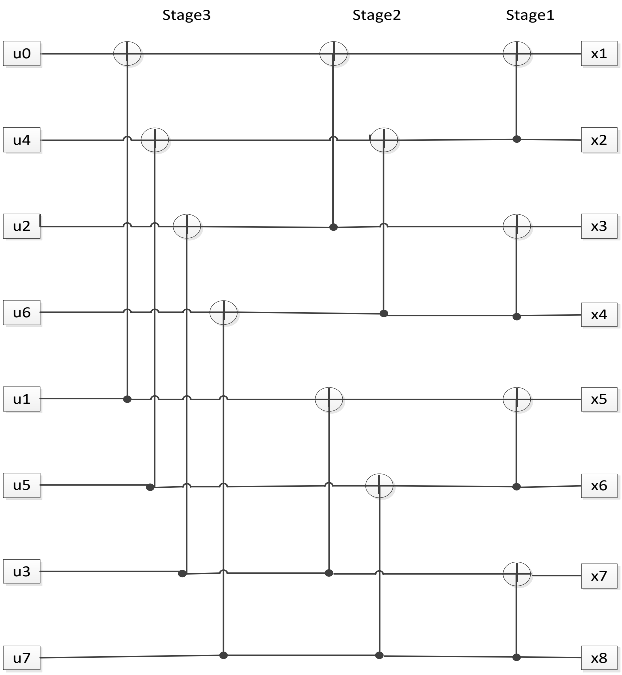

Figure 3. Coding process when $N = 8$

$$u_0(000);\ u_1(001); u_2(010); u_3(011);$$
$$u_4(100); u_5(101); u_6(110); u_7(111);$$

The representation happens to be the binary representation of subscript. For example, the bit u_0 has gone through three processes 0, so it is represented as "000", this happens to be the binary representation of the subscript "0".

C. Complementarity of Bhattacharyya Parameter

As showed in formula (8), the variation y is a function of the current Bhattacharyya parameter value x and the function is symmetric about $x = 0.5$. Suppose we have two x values: x_1,x_2,if they are symmetric about 0.5, then their variations after a polarization process are equal. Furthermore, the range of Bhattacharyya parameter value is [0,1], so we call the symmetry of the variation as the complementarity of Bhattacharyya parameter.

1) Calculating Bhattacharyya Parameter of Complementary Channel

We can calculate Bhattacharyya parameters of polarized subchannels whose original channels erasure probability are complementary according to complementarity.

For example, under the same code length, if we know the calculation result when the channel Bhattacharyya parameter is 0.1, we can get the calculation result when the channel Bhattacharyya parameters is 0.9 according to the complementarity. IF $N = 8$ and the channel erasure probability is 0.1, we can calculate that:

$$Z_1[u_{0(000)}] = 0.5695;$$
$$Z_1[u_{1(001)}] = 0.1183;$$
$$Z_1[u_{2(010)}] = 0.0709;$$
$$Z_1[u_{3(011)}] = 0.0013;$$
$$Z_1[u_{4(100)}] = 0.0309;$$
$$Z_1[u_{5(101)}] = 0.0004;$$
$$Z_1[u_{6(110)}] = 0.00019;$$
$$Z_1[u_{7(111)}] = 0.00000001;$$

Then, according to complementarity, we can get:

y_0: the variation after "process 0" when the Bhattacharyya parameter is 0.1

y_1: the variation after "process 1" when the Bhattacharyya parameter is 0.9,

$y_0 = y_1$. So

$$Z[u_{0(000)}] = 0.9 + (0.1 - Z_1[u_{0(111)}]) = 0.9999;$$
$$Z[u_{1(001)}] = 0.9 + (0.1 - Z_1[u_{0(110)}]) = 0.9998;$$
$$Z[u_{2(010)}] = 0.9 + (0.1 - Z_1[u_{0(101)}]) = 0.9996;$$
$$Z[u_{3(011)}] = 0.9 + (0.1 - Z_1[u_{0(100)}]) = 0.9691;$$
$$Z[u_{4(100)}] = 0.9 + (0.1 - Z_1[u_{0(011)}]) = 0.9987;$$
$$Z[u_{5(101)}] = 0.9 + (0.1 - Z_1[u_{0(010)}]) = 0.9291;$$
$$Z[u_{6(110)}] = 0.9 + (0.1 - Z_1[u_{0(001)}]) = 0.8817;$$
$$Z[u_{7(111)}] = 0.9 + (0.1 - Z_1[u_{0(000)}]) = 0.4341.$$

when the channel erasure probability is 0.9.

This is not an individual case. As long as the Bhattacharyya parameters of the two channels are complementary, the Bhattacharyya parameters of the polarized sub-channel can be quickly calculated according to the complementarity. For example, if we know the Bhattacharyya parameters of the polarized sub-channel when the Bhattacharyya parameter of the initial channel is 0.31, the calculation results can be quickly

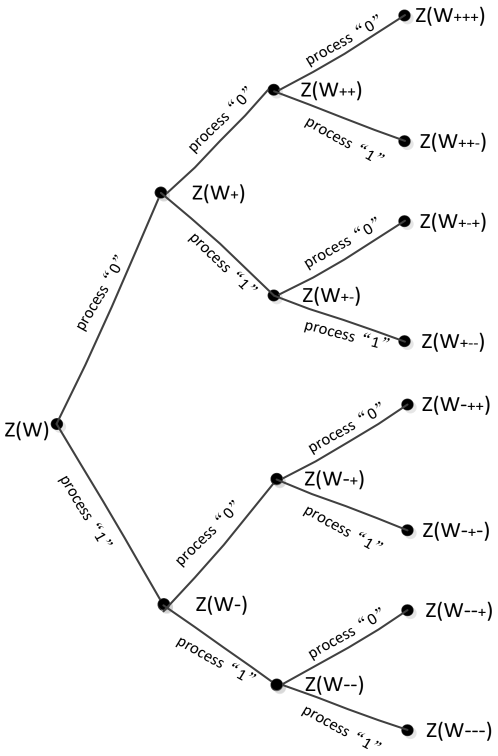

Figure 4. Changes of Bhattacharyya parameters

obtained when the Bhattacharyya parameter of initial channel is 0.69.

2) Calculating the Bhattacharyya Parameter under a Certain Channel Erasure Probability

Changes of Bhattacharyya parameters during encoding are shown in Fig4. The subscript "+" indicates the addition of Bhattacharyya parameters, while the subscript "-" indicates the deduction of the Bhattacharyya parameters. Therefore, the Bhattacharyya parameters of the polarization channel can be calculated according to the variation proposed. Assuming the Bhattacharyya parameter of channel is Z, then,

N=2,

$$u_0: Z_+ = \mathrm{Z} + \mathrm{y}(\mathrm{Z})$$
$$u_1: Z_- = \mathrm{Z} - \mathrm{y}(\mathrm{Z})$$

N=4,

$$u_{0(00)}: Z_{++} = Z_+ + y(Z_+),$$
$$u_{1(01)}: Z_{+-} = Z_+ - y(Z_+),$$
$$u_{2(10)}: Z_{-+} = Z_- + y(Z_-),$$
$$u_{3(11)}: Z_{--} = Z_- - y(Z_-);$$

We use the above method to calculate Bhattacharyya parameters, which can reduce multiplication and further decrease calculation complexity. For example, if $N = 8$and the channel Bhattacharyya parameter is $\epsilon(\epsilon \neq 0.5)$, according to the original method, 14 multiplications and 7 additions are required, but according to the algorithm proposed above, only 7 multiplications and 21 additions are needed to get the Bhattacharyya parameters of the polarized subchannel.

TABLE I. OPERATION TIMES

	multiplication	**addition**
Original method	2(*N*-1)	*N*-1
Proposed method	0	*N*

TABLE II. OPERATION TIMES

	multiplication	**addition**
Original method	2(*N*-1)	*N*-1
Proposed method	*N*-1	*3(N-1)*

Specially, if the channel Bhattacharyya parameter is 0.5, then only half the Bhattacharyya parameters of N channels need to be calculated and the remaining channels can be obtained by complementarity. For example, if $N = 8$ and the channel Bhattacharyya parameters is 0.5, we can calculate

$$\mathrm{Z}[u_{0(000)}] = 0.9961;$$
$$\mathrm{Z}[u_{1(001)}] = 0.8789;$$
$$\mathrm{Z}[u_{2(010)}] = 0.8086;$$
$$\mathrm{Z}[u_{3(011)}] = 0.3164;$$

According to complementarity,

$$\mathrm{Z}[u_{4(100)}] = 0.5 + (0.5 - \mathrm{Z}[u_{3(011)}]) = 0.6836$$
$$\mathrm{Z}[u_{5(101)}] = 0.5 + (0.5 - \mathrm{Z}[u_{2(010)}]) = 0.1914$$
$$\mathrm{Z}[u_{6(110)}] = 0.5 + (0.5 - \mathrm{Z}[u_{1(001)}]) = 0.1211$$
$$\mathrm{Z}[u_{7(111)}] = 0.5 + (0.5 - \mathrm{Z}[u_{0(000)}]) = 0.0039$$

The operation can be reduced by half according to the complementary.

D. Complexity Analysis

On the one hand, when calculating the Bhattacharyya parameters about the complementary channel, only a simple addition operation is needed. If code length is N, the operation times of the two calculation methods are shown in Table I.

On the other hand, when calculating the channel Bhattacharyya parameter under a certain channel erasure probability, the complementarity we proposed can convert the multiplication operation to the addition operation. If code length is N, the operation times of the two calculation methods are shown in Table II.

In particular, if the channel Bhattacharyya parameter is 0.5, only half Bhattacharyya parameters of N channels need to be calculated, which can reduce the computational complexity. Therefore, our method can realize the simplified calculation of Bhattacharyya parameters.

IV. CONCLUSION

Code construction is an important step in the study of polar codes. For BEC, the commonly used code construction method is the Bhattacharyya parameter method. It means to select the bits whose Bhattacharyya parameter is smaller to transmit information bits. In this paper, we discovered the inherent law of Bhattacharyya parameter that can be used to simplify calculation. Furthermore, we can calculate easily Bhattacharyya parameters of complementary channel erasure probability according to complementarity and complete the selection of information bits and frozen bits.

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